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A Noncommutative Full Completeness Theorem (Extended Abstract)

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Abstract

We present a full completeness theorem for the multiplicative fragment of a variant of noncommutative linear logic known as *cyclic linear logic (CyLL)*, first defined by Yetter. The semantics is obtained by considering dinatural transformations on a category of topological vector spaces which are invariant under certain actions of a noncocommutative Hopf algebra, called the *shuffle algebra*. Multiplicative sequents are assigned a vector space of such dinaturals, and we show that the space has the denotations of cut-free proofs in *CyLL+MIX* as a basis.

This work is a natural extension of the authors' previous work, "Linear Läuchli Semantics", where a similar theorem is obtained for the commutative logic. In that paper, we consider dinaturals which are invariant under certain actions of the additive group of integers. The passage from groups to Hopf algebras corresponds to the passage from commutative to noncommutative logic.

This is an extended abstract. A full version of this paper, with complete proofs, is in preparation.

1 Introduction

This paper is a continuation of a program initiated in [12], where a linear version of Läuchli's semantics for intuitionistic logic is presented. In that paper,

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we consider actions of the additive group of integers on a category of topological vector spaces. We associate to any sequent in Multiplicative Linear Logic (MLL) a vector space of dinatural transformations which are invariant with respect to certain such actions. We call these dinaturals *uniform*. We then show that this vector space has as basis the denotations of cut-free proofs of the sequent in the theory $MLL + MIX$. Thus we obtain a *full completeness theorem* in the sense of [2]: our semantics consists entirely of (linear combinations of) denotations of proofs.

It was observed at the end of that paper that this semantics might be expanded to noncommutative logics by replacing groups with *Hopf algebras*. In [11], the representation theory of Hopf algebras is presented as a unifying framework for the analysis of a number of variants of linear logic. By varying the Hopf structure, one obtains models of the commutative, fully noncommutative, cyclic or braided variants. Thus, choosing a Hopf algebra corresponds abstractly to specifying the structural rules of a theory. This is summarized in the following chart (the terminology is explained in [11]):

Theory	Hopf Structure
commutative	Δ cocommutative
braided	quantum group
noncommutative	Δ noncocommutative, S invertible
cyclic	Δ noncocommutative, $S^2 = id$

The relevance of Hopf algebras is further suggested by the conservativity theorem, Theorem 11.7 of [12], which says that every dinatural which is uniform with respect to the integers is also uniform with respect to arbitrary cocommutative Hopf algebras. Thus by considering general Hopf algebras, it seemed plausible that one could obtain such theorems for noncommutative logics. The full completeness theorem we present here strengthens this analogy, and suggests a general theory which we hope to explore in the future.

The particular variant of linear logic that we will work with is the *cyclic linear logic (CyLL)* of Yetter [31]. This variant is obtained by adding the cyclic exchange rule to the fully noncommutative logic of [3]. The corresponding version of proof net is also described in [31]. This theory has subsequently been used substantially by Retoré in his work on linguistics [24].

The Hopf algebra which provides our semantics is an example of the incidence algebras of [20,25]. It is also refered to as a *shuffle algebra* in [9], which is the name we have chosen to use. Given a sequent in linear logic, we assign a vector space of dinaturals which are uniform with respect to this Hopf algebra, and show that it is generated by the denotations of proofs in the cyclic fragment.

Nonsymmetric monoidal categories which arise from Hopf algebras have recently become important in quantum physics [23]. Since linear logic is a natural vehicle for describing free monoidal categories [10], then modifying the structural rule of exchange should be the logical analogue of the *quantiza-*

tion process discussed in these references. This suggests for example a logical interpretation of theorems such as the various *Tannaka-Krein* theorems described in [23,30].

The particular Hopf algebra chosen is of independent interest in several fields. In the theory of distributed and concurrent computation, an important notion is that of interleaving or merging of input streams of data. Benson [9] observed that this process has a natural algebraic structure, which led him to consider the shuffle algebra. Such structures also arise in a fundamental way in combinatorics [20,25], as such Hopf algebras provide an algebriac framework for the study of generating functions. Connections to combinatorics are further established *via* Joyal's notion of *species* [21], a functorial framework for analyzing generating functions. Species were then generalized and given a Hopf-algebraic interpretation by Schmitt in [26]. Thus the representation theory of such structures should have important consequences for both these subjects. An overview of the applications of Hopf algebras to various branches of mathematics is given by Hazewinkel in [19].

2 Cyclic Linear Logic

Yetter proposes *cyclic linear logic (CyLL)* in [31]. He presents a posetal semantics, which he calls *Girard quantales* and presents a completeness theorem, similar to the phase space completeness theorem of [15]. The logic $CyLL$ can be viewed as noncommutative linear logic, with a single negation and the following weak exchange rule:

$$\textit{Cyclic Exchange} \qquad \frac{\vdash \Gamma}{\vdash \sigma(\Gamma)} \quad \text{for any cyclic permutation } \sigma \text{ of } \Gamma.$$

Given the nature of the exchange rule for this fragment, it is natural to represent the formulas of a sequent as lying on the perimeter of a circle, or as labelling radial lines on a disk. Since we will only consider such structures up to a rotation, then we will not need any explicit representation of the cyclic exchange rule [31,24]. It is possible to represent nets by an inductive procedure analogous to that of the commutative case [15,13,14].

3 Hopf algebras and Representations

3.1 Algebras and Coalgebras

We assume that the reader is familiar with the notion of Hopf algebra. In this section we give a quick summary of the representation theory of Hopf algebras. For suitable introductions, see [1,28,19].

Definition 3.1 Given a Hopf algebra H, a *module* over H is a vector space V, equipped with a $\mathbf{k}$-linear map [2] , called an H-*action* $\rho\colon \mathsf{H}\otimes V \to V$ such that:

[2] We will assume throughout this paper that $\mathbf{k}$ is a discrete field of characteristic 0

$$\begin{tikzcd} \mathsf{H}\otimes\mathsf{H}\otimes V \arrow[r, "id\otimes\rho"] \arrow[d, "m\otimes id"] & \mathsf{H}\otimes V \arrow[d, "\rho"'] \\ \mathsf{H}\otimes V \arrow[r, "\rho"] & V \end{tikzcd} \qquad \begin{tikzcd} V & \mathsf{H}\otimes V \arrow[l, "\rho"'] \\ & k\otimes V \arrow[u, "\eta\otimes id"'] \arrow[ul, "\cong"] \end{tikzcd}$$

If (V,ρ) and (W,τ) are modules, then a map of modules, sometimes called an H-map, is a $\mathbf{k}$-linear map $f\colon V\to W$ such that:

$$\begin{tikzcd} \mathsf{H}\otimes V \arrow[r, "id\otimes f"] \arrow[d, "\rho"'] & \mathsf{H}\otimes W \arrow[d, "\tau"'] \\ V \arrow[r, "f"] & W \end{tikzcd}$$

We thus obtain a category $\mathcal{MOD}(\mathsf{H})$. We will generally denote an H-action by concatenation, e.g. $\rho(h\otimes v)=hv$, and then the above diagram can be expressed by saying that $f(hv)=hf(v)$.

The above definition is a straightforward generalization from group representations; indeed, the latter arises as the special case $\mathsf{H}=\mathbf{k}[G]$. A similar remark applies to the Hopf algebra associated to a Lie algebra [1].

If U and V are modules, then $U\otimes V$ has a natural module structure given by:

$$\mathsf{H}\otimes U\otimes V \xrightarrow{\Delta\otimes id} \mathsf{H}\otimes\mathsf{H}\otimes U\otimes V \xrightarrow{c_{23}} \mathsf{H}\otimes U\otimes\mathsf{H}\otimes V \xrightarrow{\rho\otimes\rho} U\otimes V$$

Theorem 3.2 $\mathcal{MOD}(\mathsf{H})$ *is a monoidal category. If the Hopf algebra is cocommutative, then the tensor product is symmetric. The unit for the tensor is given by the ground field with the module structure induced by the counit of* H.

Definition 3.3 Given an arbitrary Hopf algebra H with bijective antipode, and two H-modules, A and B, we will define two new H-modules, $A\multimap B$ and $B\circ\!\!- A$, as follows. In both cases, the underlying space will be $A\multimap_{\mathbf{k}} B$, the space of $\mathbf{k}$-linear maps. Note that $\Delta(h)=\Sigma h_1\otimes h_2$.

The action on $B\circ\!\!- A$ is defined by:

$$(hf)(a)=\sum h_1 f(S(h_2)a) \tag{1}$$

and the action on $A\multimap B$ is defined by:

$$(hf)(a)=\sum h_2 f(S^{-1}(h_1)a) \tag{2}$$

where $\Delta(h)=\sum h_1\otimes h_2$.

The following is proved by Majid in [23]. For categorical terminology, see [11].

Theorem 3.4 *Let* H *be a Hopf algebra with bijective antipode. Then with the actions defined above,* $\mathcal{MOD}(\mathsf{H})$ *is a biautonomous category. The adjoint relation:*

$$HOM(A \otimes B, C) \cong HOM(B, A \multimap C)$$

holds whether or not the antipode is bijective. In the case of a cocommutative Hopf algebra, the two internal HOM*'s are equal.*

To obtain a $*$-autonomous category of vector spaces, we add a topological structure, due to Lefschetz [22]. The categorical structure was worked out by Barr in [5].

Definition 3.5 Let V be a vector space. A topology, τ, on V is *linear* if it satisfies the following three properties:

- Addition and scalar multiplication are continuous, when the field $\mathbf{k}$ is given the discrete topology.
- τ is hausdorff
- $0 \in V$ has a neighborhood basis of open linear subspaces.

Let $\mathcal{TVEC}$ denote the category whose objects are vector spaces equipped with linear topologies, and whose maps are linear continuous morphisms.

$\mathcal{TVEC}$ is a symmetric monoidal closed category, when $V \multimap W$ is defined to be the vector space of linear continuous maps, topologized with the topology of pointwise convergence. (It is shown in [8] that the forgetful functor $\mathcal{TVEC} \to \mathcal{VEC}$ is tensor-preserving.) Lefschetz proves that the embedding $V \to V^{\perp\perp}$ is always a bijection, but need not be an isomorphism. We then have:

Theorem 3.6 *(Barr)* $\mathcal{RTVEC}$*, the full subcategory of reflexive objects in* $\mathcal{TVEC}$*, is a complete, cocomplete* $*$*-autonomous category.*

Definition 3.7 The category $\mathcal{TMOD}(\mathsf{H})$ is defined as follows. Objects are modules (V, ρ) such that V is equipped with a linear topology, and such that the action of H on V is continuous, when H is given the discrete topology. Maps are H-maps which are also continuous. Define $\mathcal{RTMOD}(\mathsf{H})$ to be the full subcategory of reflexive objects.

The following results are presented in [11]. They are a straightforward generalization of the results of [23].

Theorem 3.8 *Suppose that* H *is a Hopf algebra with bijective antipode. Then* $\mathcal{RTMOD}(\mathsf{H})$ *is a bi-*$*$*-autonomous category. Furthermore, if* H *has an involutive antipode, i.e.* $S^2 = id$*, then* $\mathcal{RTMOD}(\mathsf{H})$ *is a cyclic* $*$*-autonomous category.*

4 Linear Läuchli Semantics

We now review the results of [12]. (Appropriate references for the theory of dinaturality are [4,17,10]). In that paper, a full completeness theorem is established for $MLL + MIX$ *via* the notion of a *uniform dinatural*.

Definition 4.1 Let F and F' be definable functors on $\mathcal{RTVEC}$. A dinatural transformation $\theta\colon F \to F'$ is *uniform* for a group G if for every $V_1, \ldots, V_n \in \mathcal{RTMOD}(G)$, the morphism $\theta_{|V_1|,\ldots,|V_n|}$ is a G-map, i.e. is equivariant with respect to the actions induced from the atoms V_i.

It is straightforward to verify that $\mathbf{Z}$–$Dinat(F, F')$ is a vector space, under pointwise operations. We call it the *space of proofs* associated to the sequent $F \vdash F'$. (Here $\mathbf{Z}$ is the additive group of integers.)

Before obtaining a full completeness theorem, we first obtained a traditional completeness theorem, which is clearly analogous to the results of [18].

Theorem 4.2 (Completeness) *Let $M \vdash N$ be a balanced binary sequent. If the unique cut-free proof structure associated to $M \vdash N$ is not a proof net for the theory $MLL + MIX$, then $\mathbf{Z} - Dinat(M, N)$ is a zero dimensional vector space.*

The key lemma in extending this result to a full completeness theorem is:

Lemma 4.3 *Let M, N be MLL formulas. If $\mathbf{Z} - Dinat(M, N)$ has dimension greater than 0, then the sequent $M \vdash N$ is balanced.*

Now that we see that only balanced sequents need be considered, we establish the first form of full completeness:

Theorem 4.4 (Full Completeness for Binary Sequents) *Suppose that the sequent $M \vdash N$ is binary, then $\mathbf{Z} - Dinat(M, N)$ is zero or 1-dimensional, depending on whether its uniquely determined proof structure is a net. In the latter case, every $\mathbf{Z}$-dinatural is a scalar multiple of the denotation of the unique cut-free proof net.*

Note that to any balanced sequent, say $M \vdash N$, we can assign a set of sets of axiom links. This assignment determines a finite list of binary sequents of which $M \vdash N$ is a substitution instance. Suppose this list is: $M_1 \vdash N_1, M_2 \vdash N_2, \ldots$. (The list must be finite.) We define a new vector space, called the *associated binary space* for the sequent $M \vdash N$.

$$\mathcal{ABS}(M, N) = \coprod_i \mathbf{Z} - Dinat(M_i, N_i)$$

There is a canonical linear map:

$$\varphi : \mathcal{ABS}(M, N) \longrightarrow \mathbf{Z} - Dinat(M, N)$$

On basis elements, this is defined by equating literals in the sequent $M_i \vdash N_i$, or more formally, restricting which instantiations we will allow according to the pattern in $M \vdash N$.

Definition 4.5 *We call those elements of $\mathbf{Z} - Dinat(M, N)$ of the form $\varphi(\mathcal{S})$ for a (necessarily unique) $\mathcal{S} \in \mathcal{ABS}(M, N)$* diadditive.

Equivalently, a diadditive dinatural transformation is a transformation which is a linear combination of substitution instances of binary dinaturals.

We wish to mention the following lemma which was omitted from [12]. It establishes that the interpretations of distinct proofs are linearly independent, and thus our interpretation is faithful.

Lemma 4.6 *Let $\vdash \Gamma$ be a balanced, nonbinary sequent. Let $\vdash \Gamma_1, \vdash \Gamma_2, \ldots, \vdash \Gamma_n$ be the binary correct sequents which have Γ as a substitution instance. Then the set of dinatural interpretations in $\mathbf{Z} - Din(\Gamma)$ of the unique cut-free proofs of $\vdash \Gamma_1, \vdash \Gamma_2, \ldots, \vdash \Gamma_n$ are linearly independent.*

The notion of uniform diadditive dinatural transformation then gives us our full completeness theorem.

Theorem 4.7 (Full Completeness) *Let F and F' be formulas in multiplicative linear logic, interpreted as definable multivariant functors on $\mathcal{RTVEC}$. Then the vector space of diadditive $\mathbf{Z}$-uniform dinatural transformations has as basis the denotations of cut-free proofs in the theory MLL+MIX.*

We obtain the following corollary by the methods outlined in [17,10].

Corollary 4.8 *Uniform diadditive dinatural transformations compose. Thus we obtain an (indexed) $*$-autonomous category by taking as objects formulas, interpreted as multivariant functors. Morphisms will be uniform diadditive dinatural transformations.*

5 The Shuffle Algebra

We now introduce the specific Hopf algebra, known as the *shuffle algebra* with which we will work.

Let X be a set and X^* the free monoid generated by X. We denote words (= strings) in X^* by $w, w', \cdots$ and occasionally $z, z' \ldots$. Elements $x, y, \cdots \in X$ are identified with words of length 1, the empty word (= unit of the monoid) is denoted by ϵ, and the monoid multiplication is given by concatenation of strings. We denote the length of word w by $|w|$. Let $\mathbf{k}[X^*]$ be the free $\mathbf{k}$-vector space generated by X. We consider $\mathbf{k}[X^*]$ endowed with the following Hopf algebra structure [19,9]:

(i) $\mathcal{A} = \mathbf{k}[X^*]$ is an *algebra*, i.e. comes equipped with an associative $\mathbf{k}$-linear multiplication (with unit) $m : \mathcal{A} \otimes \mathcal{A} \to \mathcal{A}$:

$$w \otimes w' \quad \mapsto \quad w \bullet w' = \sum_{u \in Sh(w,w')} u \tag{3}$$

where $Sh(w, w')$ denotes the set of "shuffled" words of length $|w|+|w'|$ obtained from w and w'. Here, a *shuffle* of $w = a_1 \cdots a_m$ and $w' = a'_1 \cdots a'_n$ is a word of length $m + n$, say $w'' = c_1 \cdots c_{m+n}$ such that each of the a_i and a'_j occurs once in w'' ; moreover, within w'', a_i and a'_j occur in their original sequential order. For example, if $w = aba$ and $w' = bc$, we obtain the following set of shuffled words (where the letters from w' are underlined)

$$aba\underline{bc}, ab\underline{b}a\underline{c}, a\underline{b}ba\underline{c}, \underline{b}aba\underline{c}, ab\underline{bc}a, a\underline{b}b\underline{c}a, \underline{b}ab\underline{c}a, a\underline{bc}ba, \underline{b}a\underline{c}ba, \underline{bc}aba$$

Thus the summation $w \bullet w'$ is equal to

$$ababc + 2abbac + babac + 2abbca + babca + abcba + bacba + bcaba$$

Note that we will always denote the shuffle multiplication with $\bullet$, as opposed to the monoid multiplication, for which we use concatenation.

The unit $\eta : \mathbf{k} \to \mathcal{A}$ arises by mapping $1 \mapsto \epsilon$.

(ii) $\mathcal{A} = \mathbf{k}[X^*]$ is a *coalgebra*, i.e. comes equipped with a coassociative comultiplication (with counit) $\Delta : \mathcal{A} \to \mathcal{A} \otimes \mathcal{A}$, defined as:

$$\Delta(w) = \sum_{w_1 w_2 = w} w_1 \otimes w_2 \tag{4}$$

Note that in the equation $w_1 w_2 = w$ we are using the original monoid multiplication of X^*. The above pair $w_1 w_2$ is called a *cut* of w.

The counit $\varepsilon : \mathcal{A} \to \mathbf{k}$ is defined by:

$$\varepsilon(w) = \begin{cases} 1 \text{ if } w = \epsilon \\ 0 \text{ else} \end{cases} \tag{5}$$

Finally, there is an *antipode* defined as

$$S(w) = (-1)^{|w|}\overline{w} \tag{6}$$

where $\overline{w}$ denotes the word w written backwards.

Proposition 5.1 *$\mathcal{A} = \mathbf{k}[X^*]$ with the above structure forms a Hopf algebra with involutive antipode. Thus $\mathcal{RTMOD}(\mathcal{A})$ is a cyclic $*$-autonomous category.*

6 Full Completeness for $CyLL$

The notion of G-uniformity can be extended in an evident way to H-*uniformity*, where H is an arbitrary Hopf algebra. We call $\mathcal{A}-Dinat(F, F')$ the *space of cyclic proofs* associated to the sequent $F \vdash F'$, where $\mathcal{A}$ is the shuffle Hopf algebra. As in the commutative case, we begin by establishing that it is only necessary to consider balanced sequents. For terminology, see [12].

Proposition 6.1 *If a sequent $\vdash \Gamma$ has a nonzero $\mathcal{A}$-uniform dinatural, then it is balanced.*

Our full completeness result follows from two lemmas. They are as follows: (all sequents considered are binary.)

(i) Every $\mathcal{A}$-uniform dinatural is $\mathbf{Z}$-uniform

(ii) If $\vdash \Gamma$ is a sequent with a noncyclic list of literals, then there are no $\mathcal{A}$-uniform dinaturals.

Our full completeness result would then follow immediately. It follows from our previous work [12], summarized in section 7, that every $\mathcal{A}$-uniform dinatural is the denotation of an $MLL+MIX$ proof (up to scalar multiplication.) The second lemma then says that a $\mathbf{Z}$-uniform dinatural is $\mathcal{A}$-uniform if and only if its associated proof is a cyclic proof. Since clearly every proof in *CyLL+MIX* is a proof in *MLL+MIX*, we may conclude our full completeness theorem.

7 Main Results

By the previous discussion, we may now state:

Theorem 7.1 (Completeness for $\mathcal{A}$-Dinaturals) *Let $M \vdash N$ be a balanced binary sequent. If the unique cut-free proof structure associated to $M \vdash N$ is not a proof net for the theory $CyLL+MIX$, then $\mathcal{A}-Dinat(M,N)$ is a zero dimensional vector space.*

Theorem 7.2 (Full Completeness for Cyclic Binary Sequents) *If a sequent $M \vdash N$ is binary, then $\mathcal{A}-Dinat(M,N)$ is zero or 1-dimensional, depending on whether its uniquely determined proof structure is a cyclic net. In the latter case, every $\mathcal{A}$-dinatural is a scalar multiple of the denotation of the unique cut-free proof net.*

Theorem 7.3 (Cyclic Full Completeness) *Let F and F' be formulas in multiplicative linear logic, interpreted as definable multivariant functors on $\mathcal{RTVEC}$. Then the vector space of diadditive $\mathcal{A}$-uniform dinatural transformations has as basis the denotations of cut-free proofs in the theory CyLL+MIX.*

As usual, we are able to obtain the following corollary.

Corollary 7.4 *$\mathcal{A}$-uniform diadditive dinatural transformations compose. Thus, we obtain an (indexed) cyclic $*$-autonomous category by taking as objects formulas, interpreted as multivariant functors. Morphisms will be uniform diadditive dinatural transformations.*

8 Future Directions

The next avenue we hope to explore is extending our approach to include the additive connectives. The categories we have considered thus far are inadequate for the consideration of **MALL** in that product and coproduct are isomorphic, i.e. $\mathcal{RTVEC}$ has all finite biproducts. This problem is avoided by considering *normed vector spaces* [29], p. 96. We define a category $\mathcal{BAN}_1$ whose objects are Banach spaces, i.e. complete normed vector spaces, and whose morphisms are linear maps of norm less than or equal to 1. This is a symmetric monoidal closed category, when the tensor product is taken to be the completed projective tensor [29]. One can then add a structure analogous to the linear topology utilized here to obtain a $*$-autonomous category [6]. The resulting structures are closely related to the *mixed topologies* of [27,32]. Of course, one can also apply the Chu construction to $\mathcal{BAN}_1$ [7]. In so doing, we obtain a $*$-autonomous category of topological vector spaces in which products and coproducts no longer coincide. Explicitly, if $V, W \in \mathcal{BAN}_1$, then we have the following formulas:

$$\begin{array}{rl} \textit{Products-} & ||(v,w)|| = max\{||v||, ||w||\} \\ \textit{Coproducts-} & ||(v,w)|| = ||v|| + ||w|| \end{array}$$

These correspond to the ℓ_∞ and ℓ_1 norms respectively. Given our previous work, this seems a promising candidate for a full completeness theorem for **MALL**.

Girard, in a recent series of talks and preprint [16], has proposed the notion of a *coherent Banach space* in which the additive structure is modeled as above, and the exponentials are modeled *via* the notion of analytic functions on a Banach space. He also proposes a new version of linear sequent calculus in which the proof rules are labeled by scalars. He then shows that his semantics is sound for this theory.

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